Plasma II (PHYS-424)

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Based on lecture notes by
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Spring Semester 2025

Swiss
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News

- None

Reminder

Go to https://moodle.epfl.ch/course/view.php?id=14996 for links to notes and exercises

Plasma II

L3: Magnetohydrodynamic equilibrium configurations

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Based on the lectures notes by D. Testa

March 7, 2025

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MHD lectures

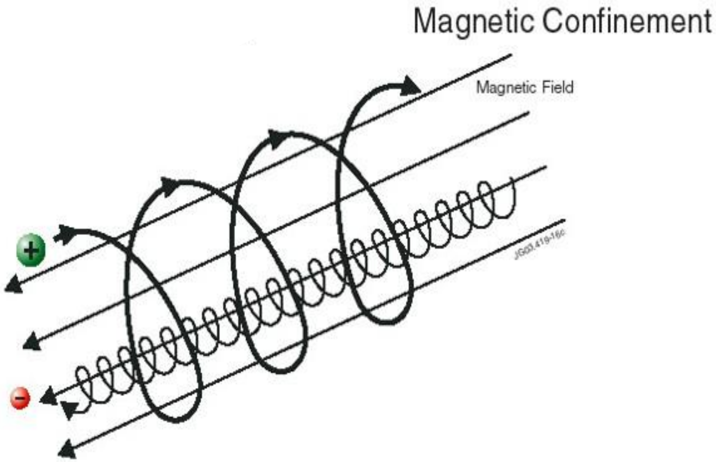
- L2: The Magnetohydrodynamic (MHD) description of a plasma
- L3: MHD equilibrium configurations of interest for magnetic confinement fusion
- L4: MHD stability and operational limits

Plasma confinement schemes for magnetic fusion

- To produce fusion in D-T plasmas we need

  $n\tau_E \sim 10^{20}\mathrm{m}^{-3}\mathrm{s}$ and $T \geq 10\mathrm{keV}$

  for break-even



- Distinguish two types of magnetic confinement schemes
  - ➢ Plasma confinement in **linear** devices → open-ended
  - ➢ Plasma confinement in **toroidal** devices ← closed-field lines

MHD equilibrium configurations

Outline

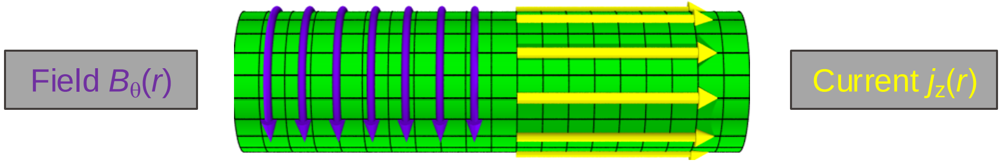
- Linear configurations
- Toroidally symmetric configurations
    - MHD equilibrium in 2-D configurations
    - The tokamak concept
- General toroidal configurations
    - The stellarator concept

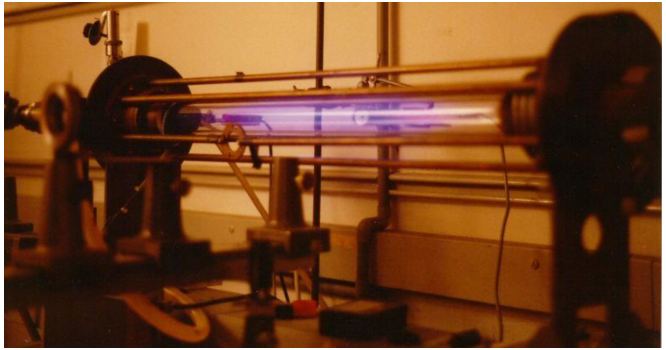
See also

- *EPFL MOOC "Plasma physics: Applications", modules 7a,b*
    - https://learning.edx.org/course/course-v1:EPFLx+PlasmaApplicationX+1T_2018/home
- Wesson, *Tokamaks* - Third Edition, Ch. 3.2-3.3, 3.9-3.12
- Freidberg, *Plasma Physics and Controlled Fusion*, Ch. 11.7

Linear confinement schemes: the Z-pinch

- Z-pinch devices: A magnetic configuration that confines a plasma using an axial current $j_Z(r)$ and a poloidal magnetic field $B_\theta(r)$ (see also L2)
  - The ZETA device (UK) was the first ever experiment to report fusion in 1957
  - The magnetic configuration of a Z-pinch can produce relaxed configuration states with τ_E>1ms
  - The Z-pinch is still used as a development path for inertial confinement



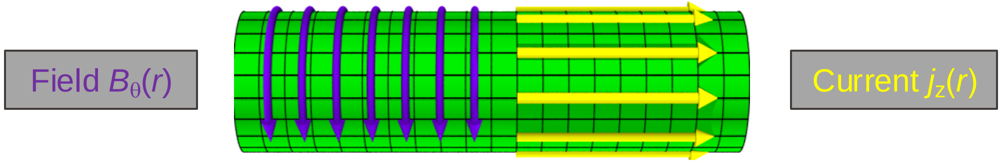


1D equilibrium

$$\frac{d}{dr}\left[p(r)+\frac{B_\theta^2(r)}{2\mu_0}\right]+\frac{B_\theta^2(r)}{\mu_0 r}=0$$

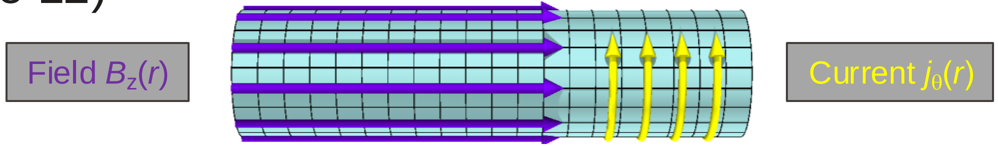
Linear confinement schemes: the Z-pinch

- World’s largest Z-pinch device: Z-machine at Sandia National Laboratory
  - Current: up to 25MA
  - Electrical power up to 80TW
  - World’s largest X-ray source with 350TW (3MJ)



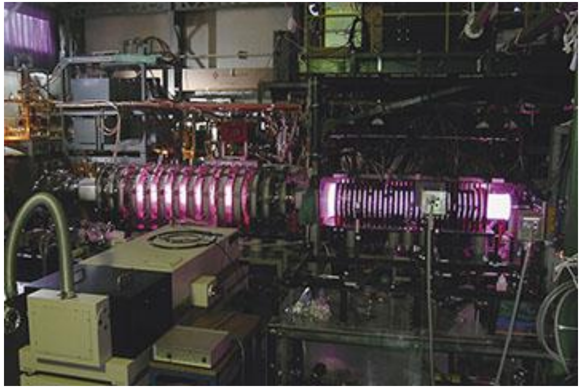
Linear confinement schemes: the Θ-pinch

- θ-pinch devices: A magnetic configuration that tries to achieve confinement of the plasma using a poloidal current $j_\theta(r)$ and an axial magnetic field $B_Z(r)$ (see also L2)
  - Still used as a development path for inertial confinement



Field $B_z(r)$

Current $j_\theta(r)$



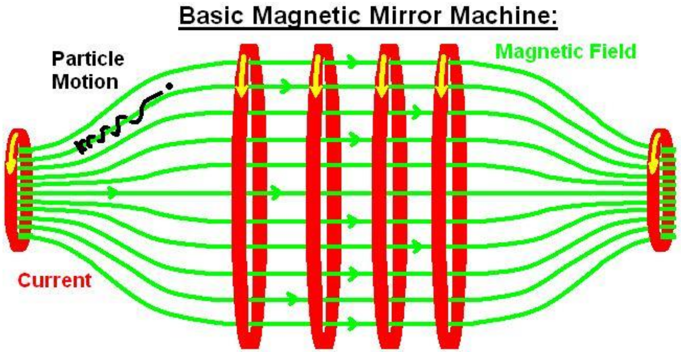
Theta pinch at Nihon University, Japan

1D equilibrium

$$\frac{B_Z^2(r)}{2\mu_0} + p(r) = \frac{B_Z^2(a)}{2\mu_0}$$

Linear confinement schemes: mirror devices

- Magnetic mirror devices (2D) → **E3-1**
  - Charged particle can be reflected from a high magnetic field region to a low magnetic field region
    - This mirror effect only occurs for particles within a limited range of velocities



- The largest mirror device ever built was the Mirror Fusion Test Facility in 1986

Linear confinement schemes: mirror devices

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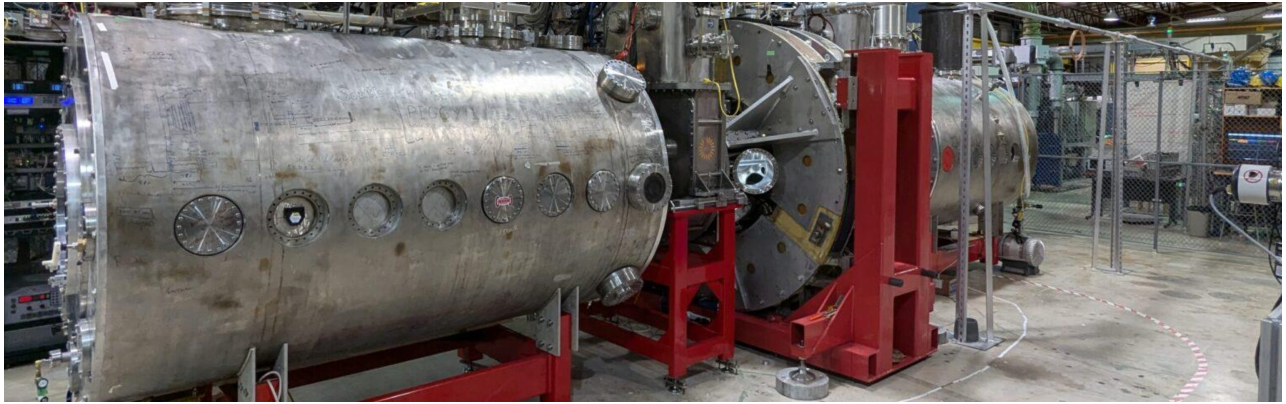


  - Wisconsin HTS Axisymmetric Mirror (WHAM) will revisit concept using state-of-the-art high-temperature superconductor (HTS) technology with field up to 17T

Magnetic confinement in linear devices: summary

- Early research experiments, simple to build
- Examples: mirror devices and pinches
- Generally very poor confinement due to losses along field lines and different classes of MHD instabilities (→ L4)
- Still used for inertial confinement experiments
- Mirror re-visited using new coil technology

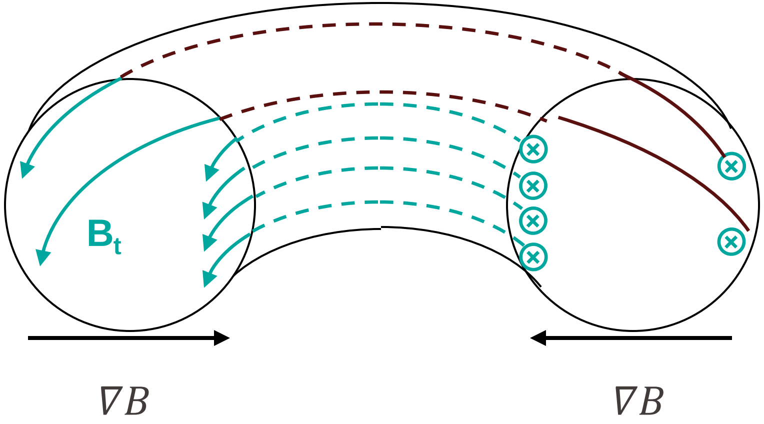
MHD equilibrium configurations

Outline

- Linear configurations
- **Toroidally symmetric configurations**
  - **MHD equilibrium in 2-D configurations**
  - The tokamak concept
- General toroidal configurations
  - The stellarator concept

Reminder: Closed field line configuration must have torus topology

- *Poincare theorem*: the only smooth 2D surface on which field lines can be covered by a non-vanishing vector field is a torus (→ L2)
- Consider a toroidal field only (toroidal theta pinch)



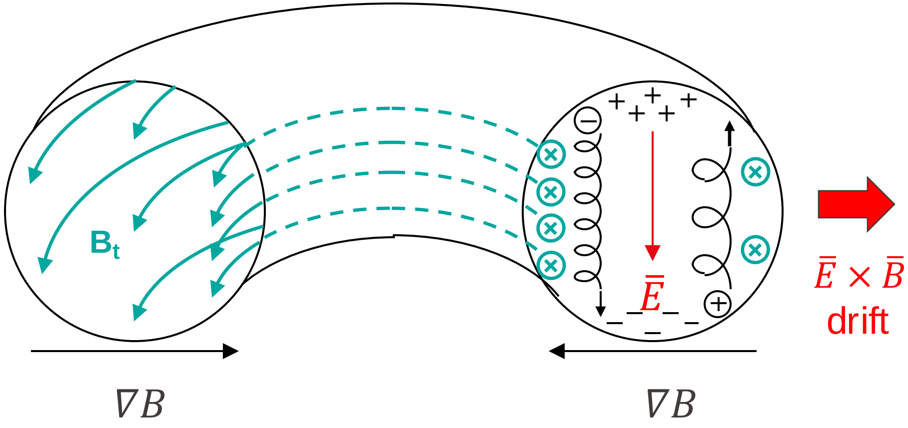
➢ 1/R dependence of B_ϕ causes $\nabla|\bar{B}|$

Magnetic field inhomogeneities lead to particle drifts (→ Introduction to Plasma Physics)

- Reminder: Guiding centre drifts (single particle picture)
  - Trajectory of charged particles in a magnetic field on time scales $\gg \tau_c$

Particle drifts in a purely toroidal field

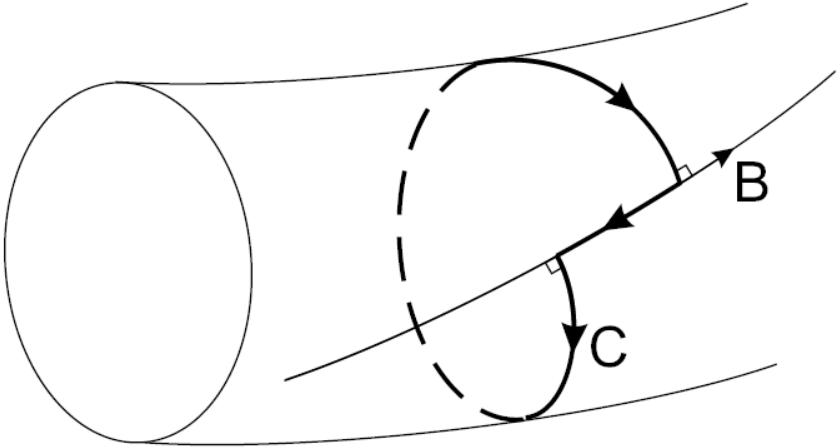
1. $\nabla|\bar{B}|$ and curvature drifts with opposite signs for ions and electrons ➔ charge separation
2. Charge separation creates vertical electric field $\bar{E}$
3. $\bar{E} \times \bar{B}$ drift leads to a collective outward motion of ions and electrons



- ➢ Confinement cannot be achieved in a pure toroidal field
- ➢ Add poloidal field (i.e. rotational transform) ➔ particles sample regions with opposite drifts and avoid charge separation

Specific case: Toroidally symmetric configuration

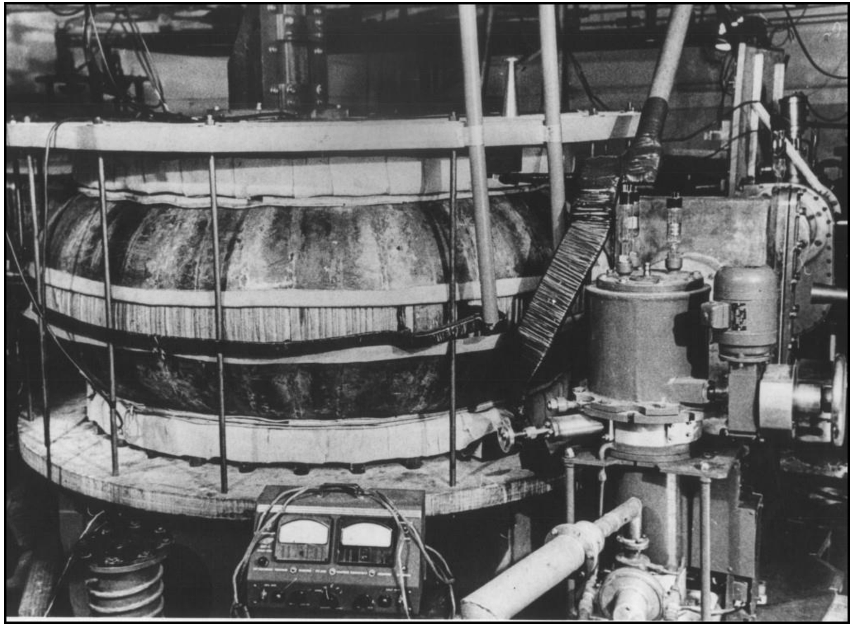
- Toroidal axisymmetric (2D) configuration with rotational transform (tokamak)



[Figure: P. Helander, EPFL seminar, 13.5.2019]

The tokamak configuration

- The tokamak (Russian acronym for ***To****roidalnaya* ***Ka****mera i* ***Ma****gnitnaya* ***K****atushka* — toroidal chamber and magnetic coils)



‘1958 configuration’ presented at the 2nd UN International Conference on the Peaceful uses of Atomic Energy, Geneva
→ later named T1 tokamak

https://www.iter.org/newsline/55/1194

The tokamak configuration

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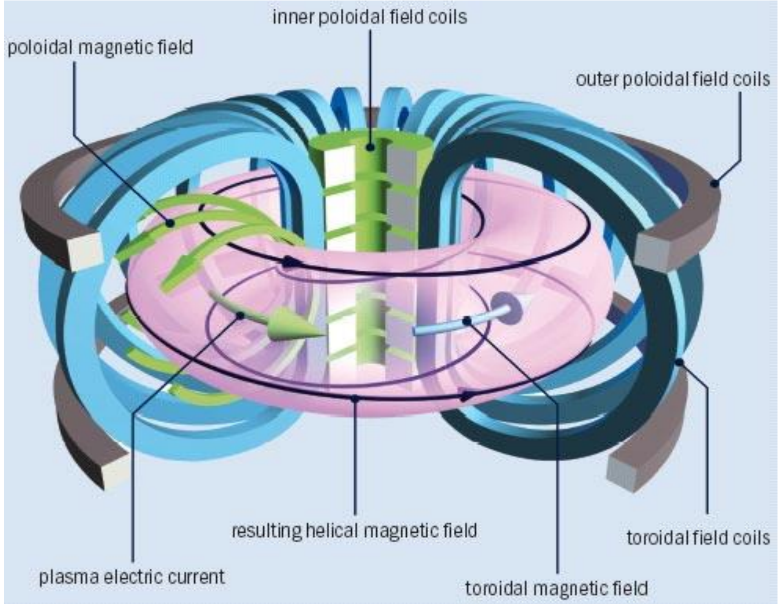
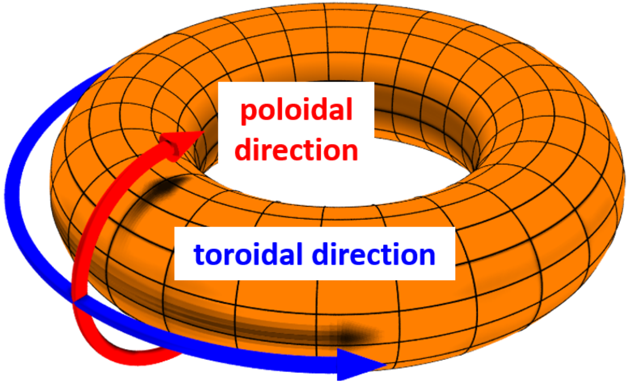


Reminder: Toroidal geometry

- A **Torus**, a cylinder with its ends folded-up, has two preferential directions
  - The ***toroidal direction (ϕ)***: the longitudinal axis of the original cylinder, defining a possible symmetry axis
  - The ***poloidal direction (θ)***: defines the plane perpendicular to the longitudinal axis of the *original* cylinder



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Def.: *Aspect ratio*

$$A \equiv \frac{\text{major radius}}{\text{minor radius}} = \frac{R_0}{a}$$

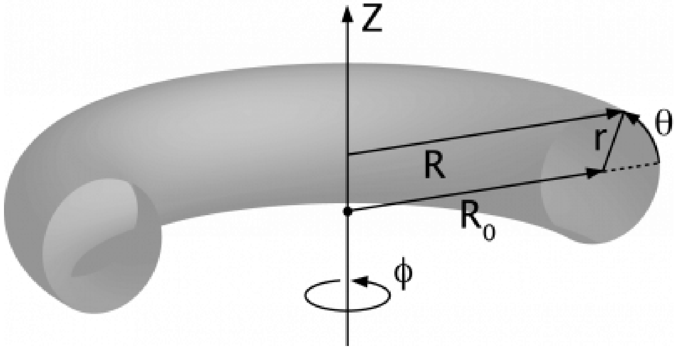
Toroidally symmetric equilibrium

- Specific case of a multi-dimensional MHD equilibrium discussed in this lecture: the toroidal axisymmetric (2D) equilibrium (tokamak equilibrium)

- How to obtain the static ($\bar{V}(\bar{x}, t) = 0$) equilibrium ($\partial/\partial t = 0$)?
  1. Use Gauss law $\nabla \cdot \bar{B} = 0$ and symmetry to constrain $\bar{B}$ components
  2. Use Ampère's law $\mu_0 \bar{J} = \nabla \times \bar{B}$ to link components of $\bar{J}$ and $\bar{B}$
  3. Replace $\bar{J}$ in MHD force balance $\nabla p = \bar{J} \times \bar{B}$

Toroidal symmetry

- Use *cylindrical coordinates* {*R*,ϕ,*Z*}
  - Note: Distinguish from *toroidal coordinates* {*r*, θ, ϕ}



- Major simplification: toroidal axisymmetric equilibrium

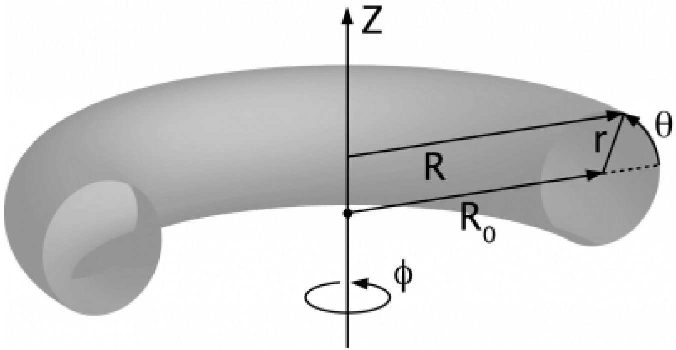
$$\rightarrow \partial/\partial\phi = 0$$

for all equilibrium quantities

Axisymmetric 2D MHD equilibrium: Flux function

- Assume toroidal symmetry $\partial/\partial\phi = 0$
  - No ϕ-dependence of equilibrium quantities including magnetic field

$$\bar{B} = \left(B_R, (R,Z), B_\phi(R,Z), B_Z(R,Z)\right)$$



- Divergence-free fields (such as $\bar{B}$) can be derived from a vector potential

$$\bar{B} = \nabla \times \bar{A}$$

  - Using $\partial/\partial\phi = 0$

$$B_R = -\frac{\partial A_\phi}{\partial Z}$$

$$B_Z = \frac{1}{R}\frac{\partial\left(RA_\phi\right)}{\partial R}$$

  - Introduce flux function $\psi = RA_\phi$

$$B_R = -\frac{1}{R}\frac{\partial\psi(R,Z)}{\partial Z}$$

$$B_Z = \frac{1}{R}\frac{\partial\psi(R,Z)}{\partial R}$$

Axisymmetric 2D MHD equilibrium: Flux function

- Note that $(\bar{B} \cdot \nabla)\psi = 0$, i.e. ψ is constant along magnetic field lines

➔ Use $\psi(R, Z)$ to label flux surfaces

Axisymmetric 2D MHD equilibrium: Flux function

- Relate flux function ψ to poloidal flux ψ_p

$$\psi_p(R,Z) \equiv \int_{S(R,Z)} \bar{B}\cdot d\bar{S} = \int_{S(R,Z)} \nabla\times\bar{A}\cdot d\bar{S}$$

$$= \int_{\partial S(R,Z)} \bar{A}\cdot d\bar{l} = 2\pi R A_\phi(R,Z) = 2\pi\psi(R,Z)$$

Def.: Poloidal flux

Axisymmetric 2D MHD equilibrium: Force balance

2. Use Ampere's law $\mu_0 \bar{\jmath} = \nabla \times \bar{B}$ to link $\bar{\jmath}$ and $\bar{B}$ (assuming axisymmetry)

$$\mu_0 j_R = -\frac{\partial B_\phi}{\partial Z} \ , \qquad \mu_0 j_\phi = \frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R} \ , \qquad \mu_0 j_Z = \frac{1}{R}\frac{\partial\left(RB_\phi\right)}{\partial R}$$

$$= -\frac{1}{R}\frac{\partial^2\psi}{\partial Z^2} + \frac{1}{R^2}\frac{\partial\psi}{\partial R} - \frac{1}{R}\frac{\partial^2\psi}{\partial R^2} = -\frac{1}{R}\Delta^*\psi$$

3. Substitute $\bar{\jmath}$ in force-balance $\nabla p - \bar{\jmath} \times \bar{B} = 0$

$$R: \qquad \mu_0\frac{\partial p}{\partial R} + \frac{B_\phi}{R}\frac{\partial\left(RB_\phi\right)}{\partial R} + \frac{1}{R^2}\frac{\partial\psi}{\partial R}\Delta^*\psi = 0$$

$$\phi: \qquad \frac{\partial\psi}{\partial R}\frac{\partial\left(RB_\phi\right)}{\partial Z} - \frac{\partial\psi}{\partial Z}\frac{\partial\left(RB_\phi\right)}{\partial R} = 0$$

$$Z: \qquad \mu_0\frac{\partial p}{\partial Z} + B_\phi\frac{\partial B_\phi}{\partial Z} + \frac{1}{R^2}\frac{\partial\psi}{\partial Z}\Delta^*\psi = 0$$

Axisymmetric 2D MHD equilibrium: Grad-Shafranov equation

3. Substitute $\bar{j}$ in force-balance $\nabla p - \bar{j} \times \bar{B} = 0$

$$R: \quad \mu_0 \frac{\partial p}{\partial R} + \frac{B_\phi}{R}\frac{\partial\left(RB_\phi\right)}{\partial R} + \frac{1}{R^2}\frac{\partial\psi}{\partial R}\Delta^*\psi = 0$$

$$\phi: \quad \frac{\partial\psi}{\partial R}\frac{\partial\left(RB_\phi\right)}{\partial Z} - \frac{\partial\psi}{\partial Z}\frac{\partial\left(RB_\phi\right)}{\partial R} = 0$$

$$Z: \quad \mu_0 \frac{\partial p}{\partial Z} + B_\phi\frac{\partial B_\phi}{\partial Z} + \frac{1}{R^2}\frac{\partial\psi}{\partial Z}\Delta^*\psi = 0$$

$$\nabla\psi \times \nabla\left(RB_\phi\right) = 0$$

RB_ϕ constant on flux surfaces

→ Introduce $F(\psi)= RB_\phi$

- Use that *p* and *F* are flux functions

$$\mu_0 R^2 \frac{\partial p}{\partial\psi} + F\frac{\partial F}{\partial\psi} + \Delta^*\psi = 0$$

Grad-Shafranov equation

Axisymmetric 2D MHD equilibrium: Grad-Shafranov equation (cont.)

- The **Grad-Shafranov** equation for flux function $\psi(R,Z)$

$$\Delta^*\psi(R,Z) = -\mu_0 R^2 \frac{\partial p\big(\psi(R,Z)\big)}{\partial\psi} - F\big(\psi(R,Z)\big)\frac{\partial F\big(\psi(R,Z)\big)}{\partial\psi}$$

  - with $\Delta^*\psi = \frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2}$

    and $\Delta^*\psi = -\mu_0 R\, j_\phi$

- Short-hand: $\Delta^*\psi = -\mu_0 R^2 p' - FF'$

Grad-Shafranov equation: solution

- The solution $\psi(R,Z)$, $p'(\psi)$, $FF'(\psi)$ of the Grad-Shafranov equation for 2D axisymmetric toroidal systems provides:

- Magnetic field as a function or *R* and *Z*
  - toroidal component: $B_\phi(R,Z) = F\big(\psi(R,Z)\big)/R$
  - poloidal components: $B_R(R,Z) = -(1/R)\,\partial\psi(R,Z)/\partial Z$

    $B_Z(R,Z) = (1/R)\,\partial\psi(R,Z)/\partial R$

- Plasma pressure $p(R,Z) = p\big(\psi(R,Z)\big)$

- Plasma current density
  - toroidal component: $j_\phi(R,Z) = -1/(R\mu_0)\,\Delta^*\psi(R,Z)$

    $= Rp'\big(\psi(R,Z)\big) + FF'\big(\psi(R,Z)\big)/(\mu_0 R)$

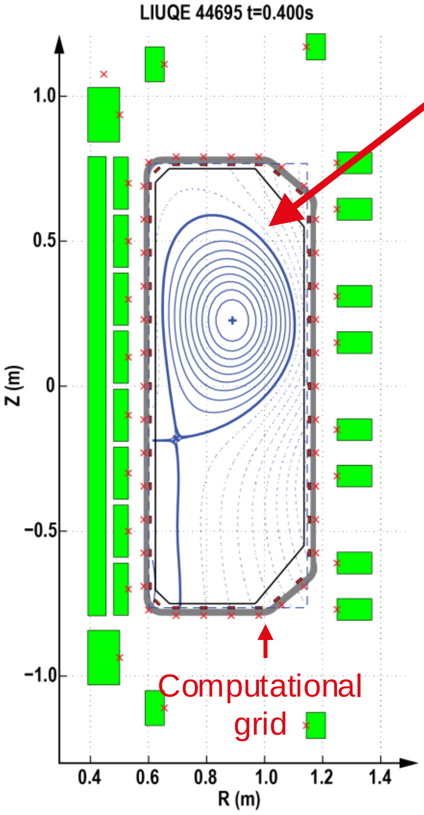
How to obtain solutions of the Grad-Shafranov equation

- **Example**

Input

Coil current measurements

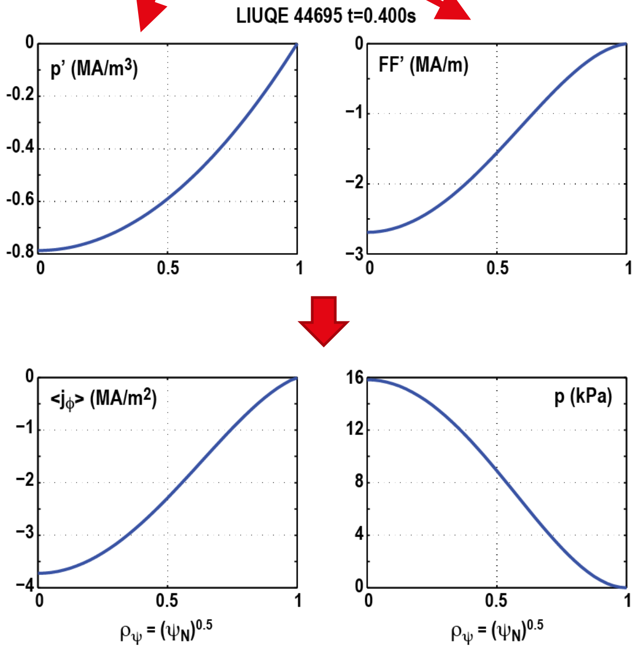
Magnetic measurements of flux and poloidal field



Computational grid

Equilibrium solver yields

- Flux contours $\psi(R,Z)$
- Profiles $p'(\psi)$ and $FF'(\psi)$



2-dimensional MHD equilibrium: summary

- Toroidicity introduces particle drifts that must be averaged out by poloidal transits (rotational transform)
- Toroidally symmetric tokamak requires a plasma current for rotational transform
- Determination of a 2-dimensional magnetic equilibrium: performed with the Grad-Shafranov equation
- Equilibrium calculated in terms of two prescribed functions $F(\psi)$ and $p(\psi)$ of the magnetic flux $\psi(R,Z)$

MHD equilibrium configurations

Outline

- Linear configurations
- **Toroidally symmetric configurations**
  - MHD equilibrium in 2-D configurations
  - **The tokamak concept**
- General toroidal configurations
  - The stellarator concept

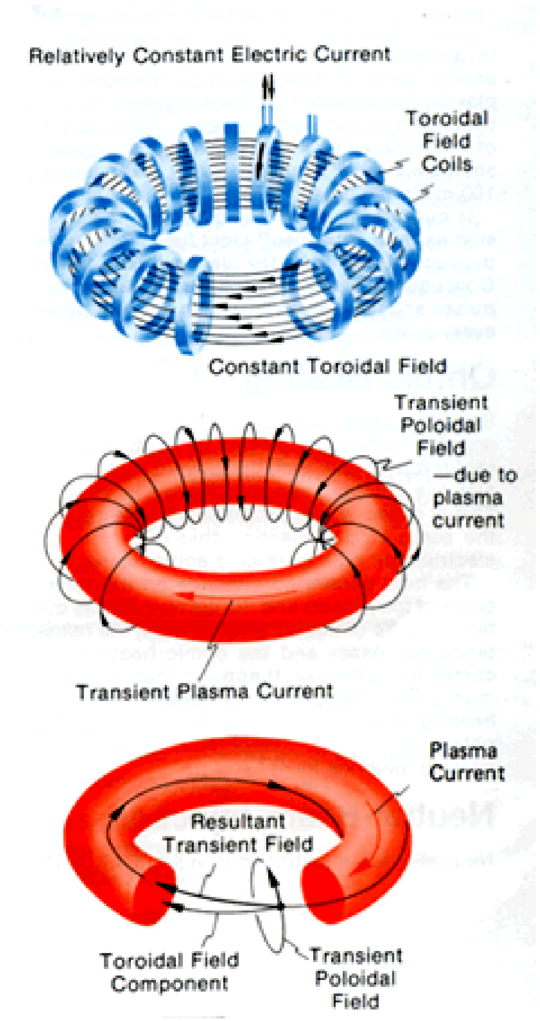
The tokamak concept

- Plasma confinement in **toroidal** magnetic fusion devices: the **tokamak**
  - Magnetic field components
  - Plasma current
  - Flux-surfaces
  - Plasma shaping
  - The safety factor
  - Typical equilibrium profiles

The tokamak concept

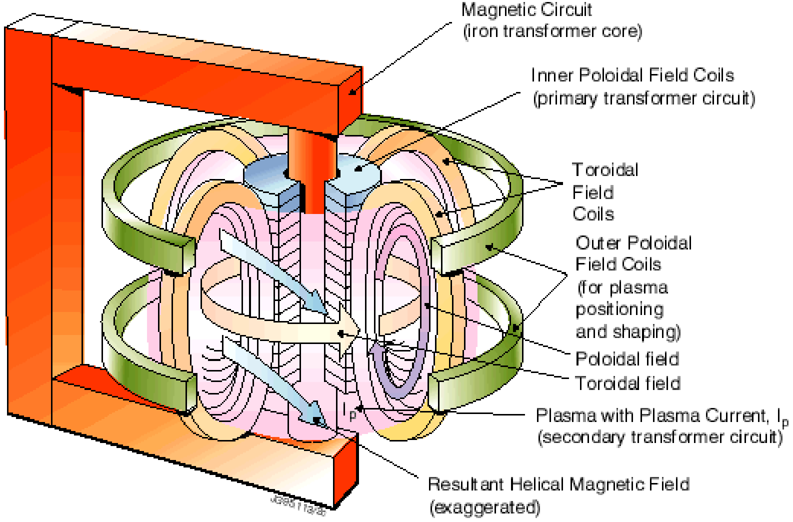
- Tokamak: superposition of different magnetic fields encloses plasma
  - ***Toroidal field***: main component, produced by external coils
  - ***Poloidal field***: generated by inducing a current in the plasma along the toroidal direction

➢ Helical field lines



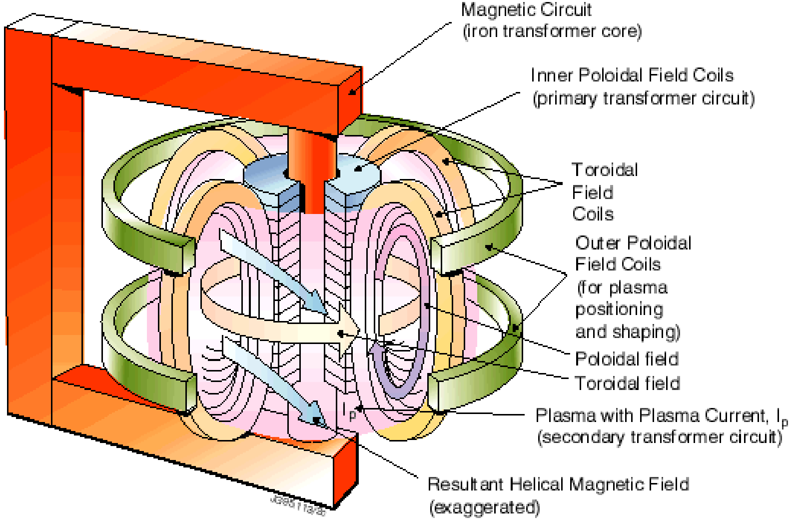
The tokamak device: TF coils

- The toroidal magnetic field (TF) is the dominant magnetic field component
- Toroidal field has a 1/*R* dependence
  - TF higher close to inner coil leg: this is the high-field side of the tokamak
  - TF lower close to outer coil leg: this is the low-field side of the tokamak



The tokamak device: ohmic transformer

- Plasma current: driven by a toroidal electric field induced by transformer action
  - Poloidal flux change in the chamber due to a current ramp in the inner poloidal field (PF) coils induces loop voltage
  - These coils act as the primary of a transformer circuit with the plasma being the secondary
  - Iron core often used for the inner PF coils to reduce power supply requirements



Toroidal force balance

- Toroidicity introduces 3 outward directed forces

a) Hoop force: Created by a toroidal current and a poloidal field

  - Stronger poloidal field on the inside leads to a net outward $\bar{j} \times \bar{B}$ force
  - Analogous to outward expansion of a current carry wire loop

b) Tire tube force: Created by the plasma pressure

  - Larger surface area *S* on the outside of an isobaric surface than on its inside leads to a net outward force
  - Analogous to larger tension on the outside of an inflated inner tube

c) 1/*R* force: Created by the 1/*R* decay of the toroidal field in the presence of poloidal currents

  - Diamagnetic* poloidal current (resulting from ∇p) leads to a larger outward force on the inside than the inward force on the outside

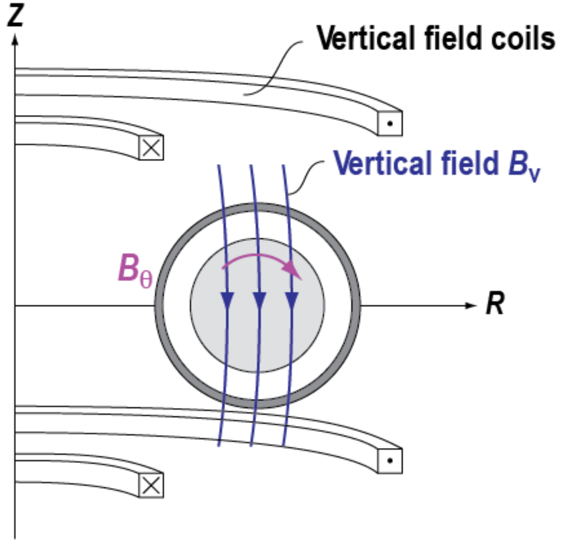
*when plasma pressure sufficiently high

The tokamak device: vertical field coils

- Exert a radial force by applying a vertical field
  - Requires vertical field coils
  - Choose direction and magnitude for $j_\phi \times B_V$ to balance outward forces

- Simple model yields
  [J. Freidberg, PP and FE, Chapter 11.7.7]

$$\frac{B_V}{B_{\theta,a}} \approx \frac{1}{4}\underbrace{\frac{a}{R_0}}_{\text{Inverse aspect ratio } \varepsilon}\left[\underbrace{l_i + l_e + 2}_{\text{Hoop force}} + \underbrace{\frac{2\mu_0\langle p\rangle}{B_{\theta,a}^2}}_{\text{Tire tube force}} + \underbrace{\frac{B_{\phi,a}^2 - \langle B_\phi^2\rangle}{B_{\theta,a}^2}}_{1/R \text{ force}}\right]$$



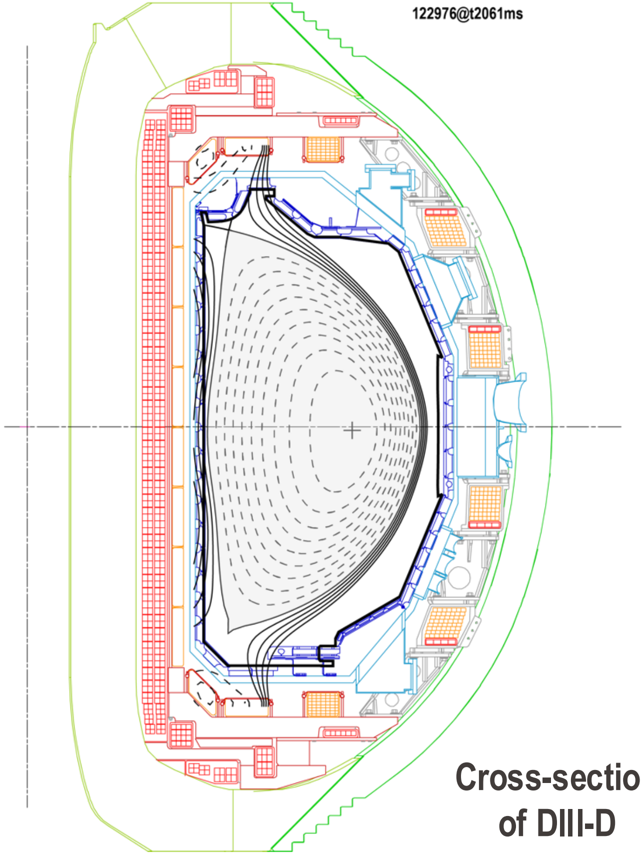
[Figure adapted from J. Freidberg, PP and FE]

Flux-surface geometry: Shafranov shift

- Flux-surfaces: contours of constant-ψ in the poloidal plane
  - Flux surfaces form a set of nested contours

- Magnetic axis (R_{mag}, Z_{mag}): extremum of the flux
  - Shafranov shift Δ: outward shift of R_{mag} with respect to the geometric centre of the plasma

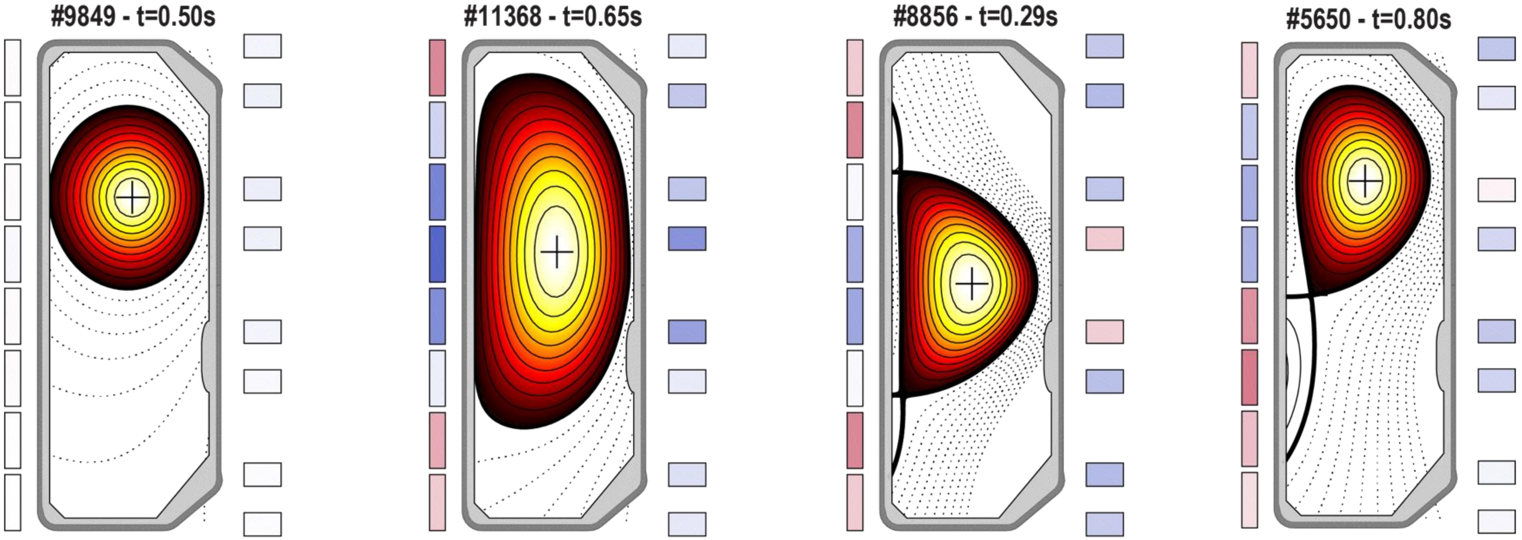
- Flux-surfaces more closely spaced on the LFS of the plasma



Cross-section of DIII-D

Flux surface geometry: plasma shaping

- Use poloidal field coils for plasma shaping



Circular

$\kappa = 1.2$
$\delta = 0.0$

Elongated

$\kappa = 2.3$
$\delta = 0.3$

Triangular

$\kappa = 1.6$
$\delta = 0.8$

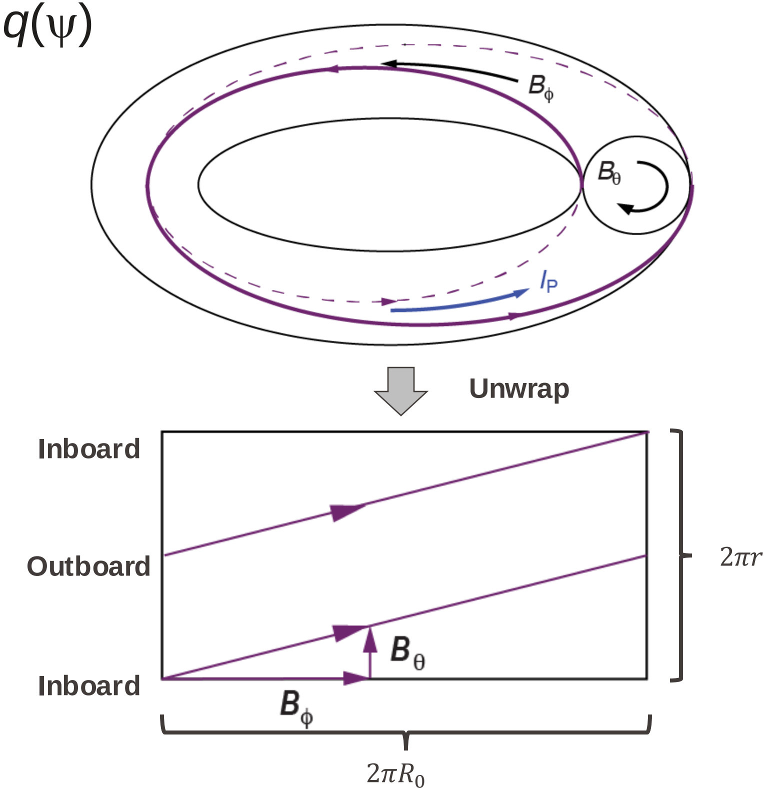
Diverted

Cross-sections of TCV

Magnetic field line geometry

- Safety factor: defined as a flux-function $q(\psi)$
  - Consider large aspect ratio limit $R_0/a >> 1$ and a circular plasma

Reminder: Safety factor



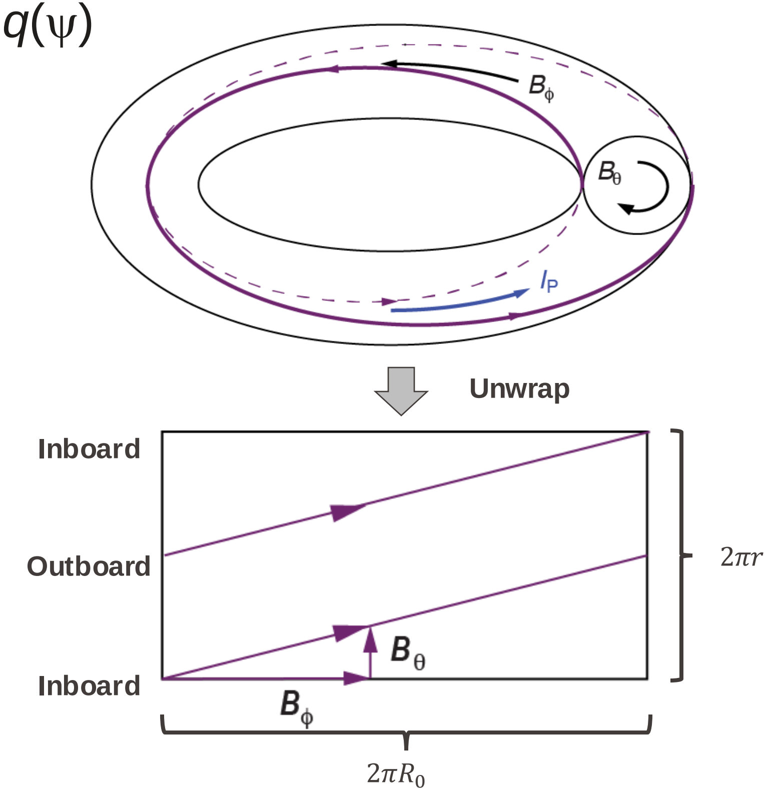
Magnetic field line geometry

- Safety factor: defined as a flux-function $q(\psi)$
  - Consider large aspect ratio limit $R_0/a >> 1$ and a circular plasma

$$q(r) = \frac{rB_{\phi,0}}{R_0 B_\theta(r)}$$

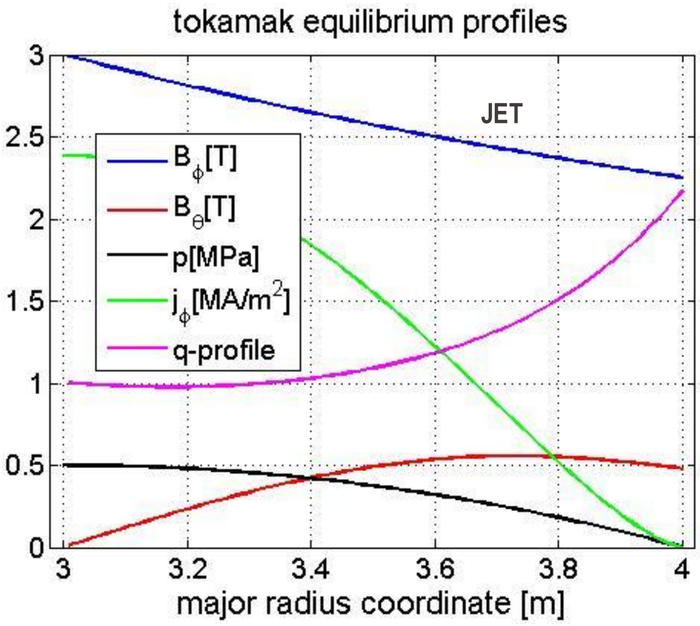
$$= \frac{2\pi r^2 B_{\phi,0}}{\mu_0 R_0 I(r)}$$

  - Rational q-surfaces: $q_{res}=m/n$
  - Perturbations can resonate with this natural periodicity of the equilibrium ➔ $q(r)$ defines plasma stability!



Typical equilibrium profiles

- Tokamak equilibrium profiles:
  2D axis-symmetric equilibrium
  - Toroidal magnetic field $B_\phi(R) \propto 1/R$
  - Poloidal magnetic field $B_\theta(R,Z)$ increases from the center, where $B_\theta(R_{mag})=0$, towards the edge
    - Typically $B_\theta/B_\phi \approx 0.1$
  - Plasma pressure $p(\psi)$ peaks on the magnetic axis
  - Toroidal current $j_\phi(R,Z)$ typically peaks on the magnetic axis



  - Safety factor $q(\psi)$ typically increases from the magnetic axis towards the edge
    - Values depend on current profile and plasma shape (**see E3-2**)

The tokamak concept: summary

- Plasma current driven by a toroidal electric field usually induced by transformer action → intrinsically pulsed
- Toroidicity leads to outward forces that must be balanced by a vertical magnetic field
- Tokamak: superposition of various magnetic fields
  - Toroidal field: dominant component, generated by external coils
  - Poloidal field: generated by inducing a toroidal plasma current
  - Vertical field: generated by external coils
- Toroidal current j_ϕ and plasma pressure p typically peak on the magnetic axis

MHD equilibrium configurations

Outline

- Linear configurations
- Toroidally symmetric configurations
    - MHD equilibrium in 2-D configurations
    - The tokamak concept
- **General toroidal configurations**
    - The stellarator concept

Stellarator vs. Tokamak

- Tokamak: a 2D toroidal device

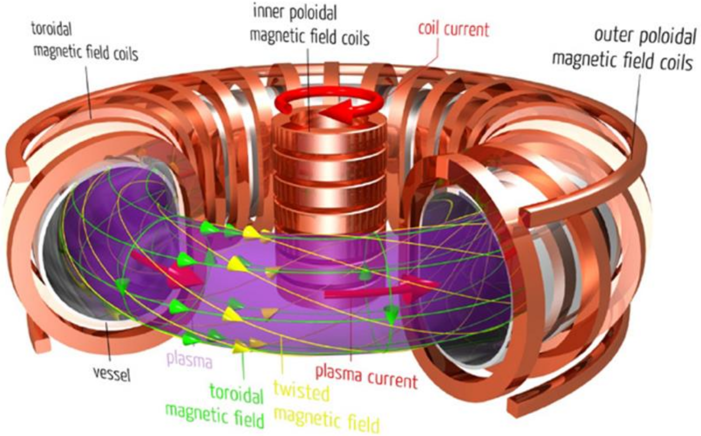
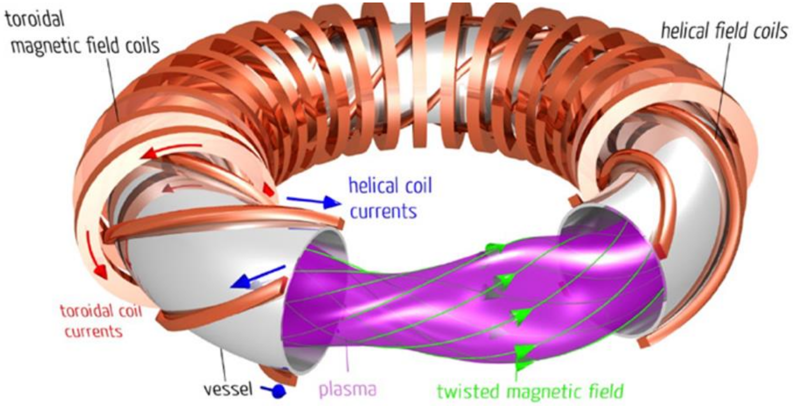


Image Credits: Max-Planck Institut für Plasmaphysik

- Stellarator: a 3D toroidal device



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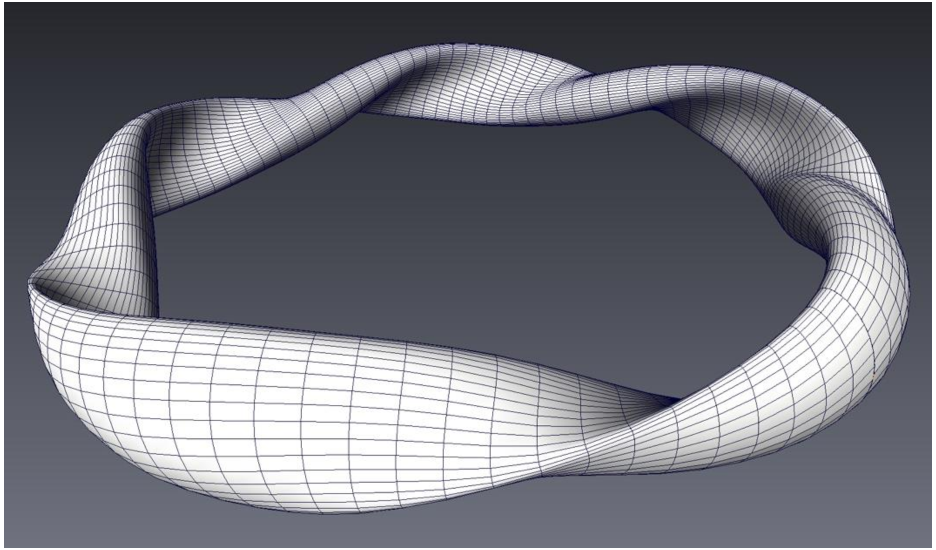
Stellarator vs. Tokamak

- 3D geometry necessary to generate rotational transform without a plasma current



[Figure: P. Helander, EPFL seminar, 13.5.2019]

Stellarator vs. Tokamak

- **Stellarator**: no externally driven plasma current, plasma confinement solely due to the externally driven magnetic field components

- **Stellarator**: toroidal equilibrium is truly non-axisymmetric in 3D ➔ all equilibrium quantities are also function of the toroidal angle ϕ

- **Tokamak**: a toroidal plasma current is used to produce a poloidal magnetic field and obtain rotational transform

- **Tokamaks**: axisymmetric system: $\partial/\partial\phi\equiv0$ for all equilibrium quantities ➔ 2D toroidal equilibrium!

From 2D to 3D

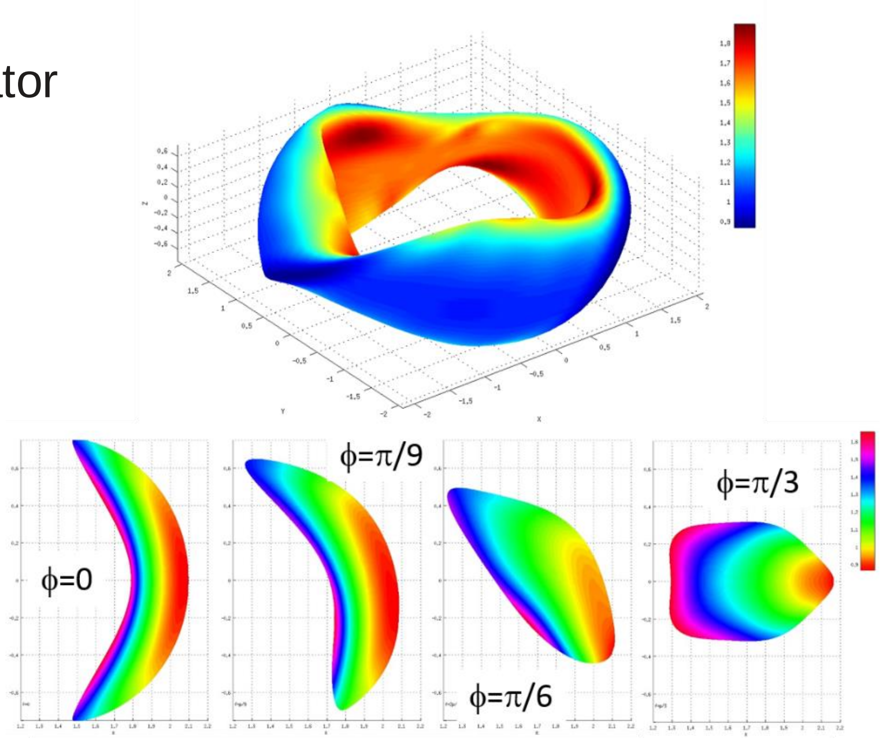
- 3D equilibrium configurations are found using a variational approach ➔ minimise the total energy of the system

$$W = \int \left(\frac{|\bar{B}|^2}{2\mu_0} + \frac{3}{2} p \right) dV$$

  - It can be shown that in the 2D limit the variational approach is equivalent to solving the Grad-Shafranov equation

Stellarator equilibrium

- **Example**: the NCSX stellarator

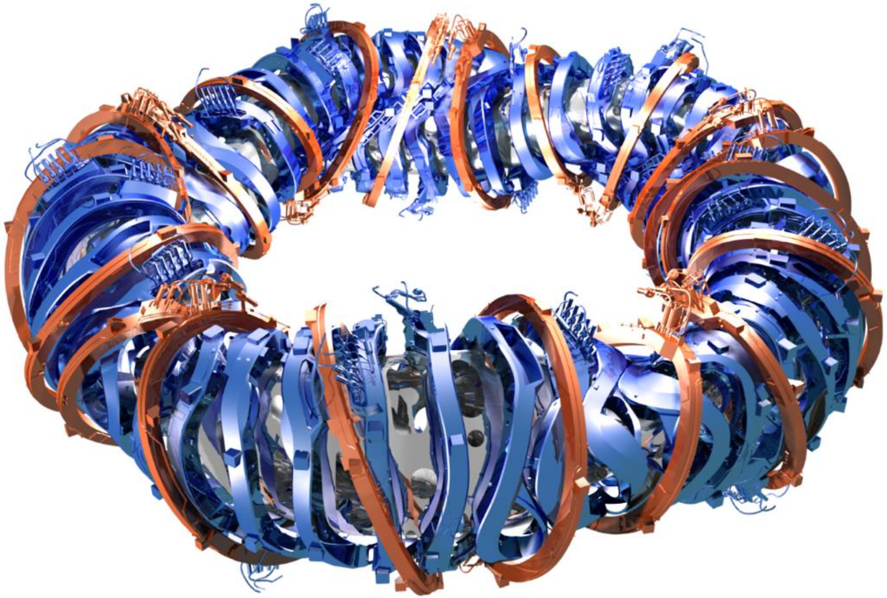


toroidal variation of equilibrium for the NCSX stellarator

Image Credits: Dr. W.A.Cooper, CRPP-EPFL

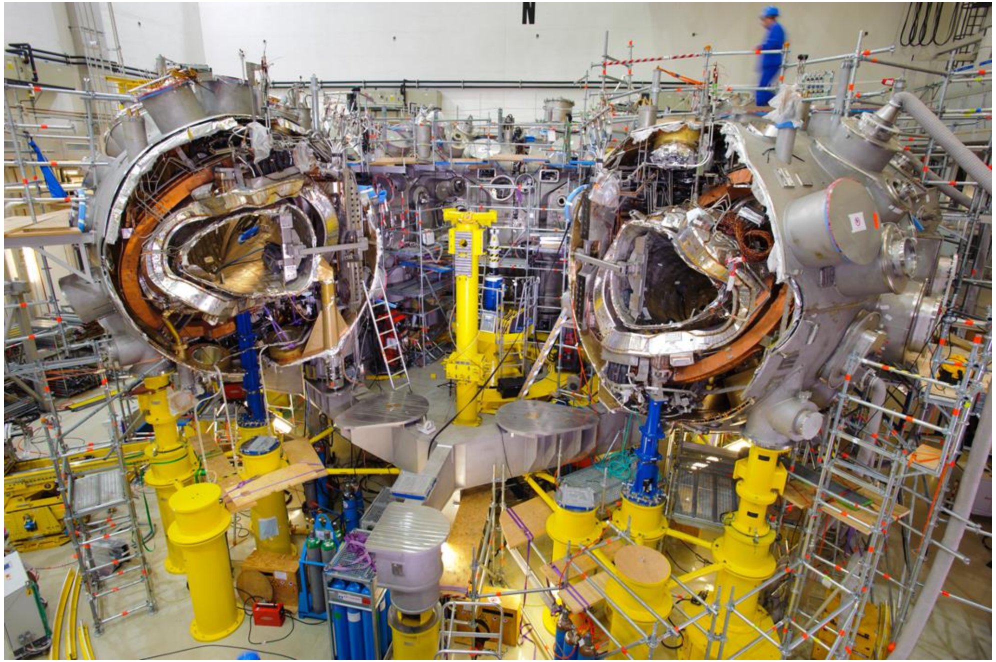
Toroidal confinement in the stellarator

- **Stellarator**: 3D magnetic field provided by 3D external coils
  - With super-conducting magnets, magnetic field can (in principle) be maintained continuously



Toroidal confinement in the stellarator

- **Stellarator**: very complicated 3D shape for super-conducting coils
  - Magnetic field must be accurately tailored to prevent energetic particles from escaping



Wendelstein 7-X (IPP-Greifswald, Germany)

- Major radius: 5m
- Minor radius: 0.53m
- Volume: 30m^3
- Magnetic field: 3T

Re-cap: Magnetohydrodynamic equilibrium configurations

- Toroidal (closed field line) configurations have superior confinement properties than linear (open ended) configurations
- Axisymmetric (2D) equilibria are described by the Grad-Shafranov equation
  - Pressure p and $F=RB_\phi$ are functions of the flux ψ
  - Need for an externally applied vertical field for toroidal force balance
- Tokamak components are toroidal field coils, inner poloidal field coils (primary of the transformer), vertical field coils (for equilibrium) and poloidal field coils (for shaping)
  - Induction of plasma current makes a tokamak intrinsically pulsed
- Stellarators generate the magnetic field (predominantly) with external coils
  - Stellarators are in principal capable for steady-state operation